ON CERTAIN AUTOMORPHISMS OF SETS OF PARTIAL ISOMETRIES

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Abstract. Under the mild condition of continuity at a single point we describe all the bijections of the set of all partial isometries on a Hilbert space which preserve the order and the orthogonality in both directions. Moreover, we present a natural analogue of Wigner's theorem on quantum mechanical symmetries for the set of all rank-1 partial isometries.

In the mathematical foundations of quantum mechanics the lattice $P(H)$ of all projections on a Hilbert space H plays a fundamental role. The whole set of projections equipped with the usual partial ordering and orthogonality represents the probabilistic aspect of the theory and the set $P_1(H)$ of all rank-1 projections with the notion of transition probability is the object of the fundamental theorem of Wigner on quantum mechanical symmetries. Clearly, the projections can be characterized as positive partial isometries. If we drop positivity, one can naturally raise several problems concerning partial isometries which are familiar in relation to projections. Probably the most fundamental results concerning the mentioned structures of projections are the description of all bijective transformations of $P(H)$ which preserve the order and the orthogonality in both directions, and Wigner's theorem (sometimes called unitary-antiunitary theorem) determining all bijective transformations of $P_1(H)$ which preserve the transition probabilities (these results can be found, for example, in [1]; also see the references therein). The aim of this paper is to present analogue results for the corresponding structures of partial isometries. These operators, we mean the partial isometries, play the same role in the relatively new theory of ternary structures of operators as the projections do in connection with the usual binary structures. The former structures are of great importance in several subareas of functional analysis, for example, in the study of the geometrical properties of operator algebras (see [2] and its references).

We begin with the notation that we use throughout. Let H be a complex separable Hilbert space. The algebra of all bounded linear operators on H is denoted by $B(H)$. The symbol $F(H)$ stands for the set of all finite rank elements of $B(H)$. An operator $U \in B(H)$ is called a partial isometry if

Date: February 1, 2008.

1991 *Mathematics Subject Classification.* Primary: 47B49.

This research was supported by the Hungarian National Foundation for Scientific Research (OTKA), Grant No. T030082, T031995, and by the Ministry of Education, Hungary, Reg. No. FKFP 0349/2000.

there is a closed subspace I_U of H called the initial space of U for which it holds that U acts as an isometry on I_U and U is zero on $I_U^{\perp}$. The range of U is called the final space of U and it is denoted by F_U. The set of all partial isometries on H is denoted by $PI(H)$. The symbol $PI_1(H)$ stands for the set of all rank-1 partial isometries on H. We introduce a partial ordering on $PI(H)$. For any $P, Q \in PI(H)$ we write $P \leq Q$ if $I_P \subset I_Q$, $F_P \subset F_Q$ and $P_{|I_P} = Q_{|I_P}$. We say that the partial isometry P is orthogonal to the partial isometry Q if $P^*Q = PQ^* = 0$. Obviously, the partial ordering introduced above coincides, when restricting to projections, with the usual order on $P(H)$. The same holds true for the orthogonality relation.

A linear norm preserving bijection on H is called a unitary operator , while any conjugate-linear norm preserving bijection on H is called an antiunitary operator.

If $x, y \in H$, then $x \otimes y$ stands for the operator defined by

$$(x \otimes y)(z) = \langle z, y \rangle x \qquad (z \in H).$$

The usual trace functional on the trace class operators is denoted by tr. Finally, the set of all complex numbers of modulus 1 is denoted by $\mathbb{T}$.

Our first result which follows describes all the bijections of $PI(H)$ which preserve the order and orthogonality in both directions.

Theorem 1. *Let H be a complex Hilbert space with $\dim H \geq 3$. Suppose that $\phi : PI(H) \to PI(H)$ is a bijective transformation which preserves the partial ordering and the orthogonality between partial isometries in both directions. If ϕ is continuous (in the operator norm) at a single element of $PI(H)$ different from 0, then ϕ can be written in one of the following forms:*

(i) *there exist unitaries U, V on H such that*

$$\phi(R) = URV \qquad (R \in PI(H));$$

(ii) *there exist antiunitaries U, V on H such that*

$$\phi(R) = URV \qquad (R \in PI(H));$$

(iii) *there exist unitaries U, V on H such that*

$$\phi(R) = UR^*V \qquad (R \in PI(H));$$

(iv) *there exist antiunitaries U, V on H such that*

$$\phi(R) = UR^*V \qquad (R \in PI(H)).$$

Proof. We assert that $P \in PI(H)$ is a rank-n partial isometry if and only if so is $\phi(P)$. By the order preserving property of ϕ we readily have $\phi(0) = 0$ which verifies our claim in the case $n = 0$. Suppose that we have the assertion for $k = 0, \dots, n$. Let P be a partial isometry on H. It is clear that P has rank $n+1$ if and only if for every $Q \in PI(H)$ with $Q \neq P$, $Q \leq P$ it follows that the rank of Q is less than or equal to n and there is such a Q whose rank is n. The order preserving property of ϕ now implies that the rank of $\phi(P)$ is $n+1$.

Our next aim is to show that ϕ is completely orthoadditive. The meaning of this property will be clear in a moment. Let $R_\alpha \in PI(H)$ be a set of pairwise orthogonal partial isometries. It is well-known that the series $\sum_\alpha R_\alpha$ is convergent in the strong operator topology and its sum R is a partial isometry. The sum $\sum_\alpha \phi(R_\alpha)$ is also a partial isometry, so there exists a $Q \in PI(H)$ such that

$$\sum_\alpha \phi(R_\alpha) = \phi(Q).$$

Since $R_\alpha \leq R$ for every α, it follows that $\phi(Q) = \sum_\alpha \phi(R_\alpha) \leq \phi(R)$. On the other hand, since $\phi(R_\alpha) \leq \phi(Q)$, we have $R_\alpha \leq Q$ for every α and we obtain that $R = \sum_\alpha R_\alpha \leq Q$. This yields $\phi(Q) = \phi(R)$. Therefore, we have

$$\sum_\alpha \phi(R_\alpha) = \phi(\sum_\alpha R_\alpha)$$

and this means the complete orthoadditivity of ϕ.

Take $3 \leq n \in \mathbb{N}$. Let M, N be n-dimensional subspaces of H. Denote by $PI(M,N)$ the set of all partial isometries on H whose initial space is a subset of M and whose final space is a subset of N. Let P be a partial isometry on H whose initial space is M and whose final space is N. Then $\phi(P)$ is a rank-n partial isometry. Let $M' = I_{\phi(P)}$ and $N' = F_{\phi(P)}$. It is easy to see that an operator $Q \in PI(H)$ belongs to $PI(M,N)$ if and only if for every $R \in PI(H)$ which is orthogonal to P it follows that R is orthogonal to Q. By the orthogonality preserving property of ϕ we deduce that ϕ maps $PI(M,N)$ onto $PI(M',N')$. Clearly, $PI(M,N)$ and $PI(M',N')$ are both isomorphic to the space $PI(H_n)$ of all partial isometries on an n-dimensional Hilbert space H_n.

Therefore, our map ϕ induces a transformation ψ on $PI(H_n)$ which has the same preserver properties as ϕ. Obviously, $\psi(I)$ is unitary. Multiplying ψ by a fixed unitary element of $B(H_n)$, we can assume that $\psi(I) = I$. This latter property implies that ψ sends projections to projections. In fact, for any partial isometry P we have $P \leq I$ if and only if P is a projection. So, ψ is a bijective transformation of the set of all projections in $B(H_n)$ which preserves the order and the orthogonality in both directions. The form of such transformations is well-known. It follows, for example, from [1] that there exists an either unitary or antiunitary operator U on H_n such that

$$\psi(P) = UPU^* \tag{1}$$

for every projection P in $B(H_n)$.

Let $\lambda \in \mathbb{T}$. We state that $\psi(\lambda I)$ is a scalar multiple of the identity. To see this, first observe that if $R \in PI(H)$ is a rank-1 partial isometry, then $\phi(\lambda R)$ is a scalar multiple of $\phi(R)$ (which scalar obviously has modulus 1). This follows from the fact that any partial isometry in $B(H)$ is orthogonal to $\phi(\lambda R)$ if and only if it is orthogonal to $\phi(R)$ and that $\phi(\lambda R)$, $\phi(R)$ are both of rank 1. Now, one can characterize the scalar operators in $PI(H_n)$ in the following way. The partial isometry $A \in PI(H_n)$ is equal to the identity

multiplied by a scalar from $\mathbb{T}$ if and only if for every rank-one projection P we have that $\lambda P \leq A$ for some $\lambda \in \mathbb{T}$. Therefore, we obtain that ψ preserves the scalar partial isometries. Consequently, there is a function $f : \mathbb{T} \to \mathbb{T}$ such that

$$\psi(\lambda I) = f(\lambda) I.$$

We know that if $P \in B(H_n)$ is any rank-one projection and $\lambda \in \mathbb{T}$, then there is a $\mu \in \mathbb{T}$ such that $\mu\psi(P) = \psi(\lambda P)$. As $\psi(\lambda P) \leq \psi(\lambda I)$, we obtain that $\mu\psi(P) \leq f(\lambda) I$. This implies that $\mu = f(\lambda)$, so we have

$$\psi(\lambda P) = f(\lambda)\psi(P)$$

for every rank-1 projection $P \in B(H_n)$ and $\lambda \in \mathbb{T}$. By the orthoadditivity of ψ (this follows form the orthoadditivity of ϕ) we infer that the previous equality holds true for every projection $P \in B(H_n)$ without any restriction on its rank.

If $R \in PI(H_n)$ is any partial isometry, then R can be written in the form $R = UP$, where U is unitary and P is a projection. Considering the transformation $Q \mapsto \psi(U)^*\psi(UQ)$ on $PI(H_n)$ and applying what we have proved above, it follows that there is a function $f_R : \mathbb{T} \to \mathbb{T}$ such that

$$\psi(\lambda R) = f_R(\lambda)\psi(R) \qquad (\lambda \in \mathbb{T}).$$

The function f_R might depend on R since the function f appearing above depends on ψ. However, we prove that $f_R = f$ for every partial isometry $R \in PI(H_n)$. This will be done below.

Let $R \in PI(H_n)$ be a rank-1 partial isometry. Pick a rank-1 projection $P \in B(H_n)$ which is orthogonal to R. By the orthoadditivity of ψ, for any $\lambda \in \mathbb{T}$ we compute

$$\psi(\lambda R + \lambda P) = f_{R+P}(\lambda)\psi(R + P) = f_{R+P}(\lambda)(\psi(R) + \psi(P)).$$

On the other hand, we have

$$\psi(\lambda R + \lambda P) = \psi(\lambda R) + \psi(\lambda P) = f_R(\lambda)\psi(R) + f(\lambda)\psi(P).$$

These imply that $f_R = f_{R+P} = f$. So we have

$$\psi(\lambda R) = f(\lambda)\psi(R) \qquad (\lambda \in \mathbb{T}) \tag{2}$$

for every rank-1 partial isometry $R \in PI(H_n)$. Since every partial isometry is the sum of mutually orthogonal rank-1 partial isometries, by the orthoadditivity of ψ we obtain that the above equality holds also for every partial isometry R on H_n.

Since for every $\lambda, \mu \in \mathbb{T}$ we have

$$f(\lambda\mu) I = \psi((\lambda\mu) I) = \psi(\lambda(\mu I)) = f(\lambda) f(\mu) I,$$

it follows that $f : \mathbb{T} \to \mathbb{T}$ is a multiplicative bijection.

We supposed that our original transformation is norm-continuous at a point $0 \neq R \in PI(H)$. We can find pairwise orthogonal rank-1 partial

isometries R_α for which $\sum_\alpha R_\alpha = R$. By the complete orthoadditivity of ϕ we obtain that

$$\sum_\alpha \phi(\lambda R_\alpha) = \phi(\lambda R).$$

As we have seen above, the operator $\phi(\lambda R_\alpha)$ is a scalar multiple of $\phi(R_\alpha)$ for every $\lambda \in \mathbb{T}$. Let λ_n be a sequence in $\mathbb{T}$ converging to 1. This implies that $\phi(\lambda_n R) \to \phi(R)$ and then we have $\phi(\lambda_n R_\alpha) \overset{n\to\infty}{\longrightarrow} \phi(R_\alpha)$ for every α. Pick an α_0 and let us suppose that $R_{\alpha_0} \in PI(M,N)$. We state that f is continuous. In fact, by (2) we can deduce that f is continuous at 1. Since f is multiplicative, we obtain that f is a continuous character of $\mathbb{T}$. It is well-known that the continuous characters of $\mathbb{T}$ are exactly the functions $z \mapsto z^n$ $(n \in \mathbb{Z})$. Since f is bijective, we obtain that f is either the identity or the conjugation on $\mathbb{T}$.

Suppose that the operator U in (1) is unitary and suppose that f is the conjugation on $\mathbb{T}$. Using the spectral theorem and the orthoadditivity of ψ, we obtain from (2) that for every unitary R in $B(H_n)$ we have

$$\psi(R) = UR^*U^*.$$

If $R \in PI(H_n)$ is arbitrary, then there is a partial isometry R' on H_n which is orthogonal to R such that $R + \lambda R'$ is unitary for every $\lambda \in \mathbb{T}$. It follows that

$$\psi(R) \le \psi(R + \lambda R') = U(R + \lambda R')^*U^* = UR^*U^* + \overline{\lambda} U R'^* U^*$$

for every $\lambda \in \mathbb{T}$. This obviously implies that $\psi(R) \le UR^*U^*$. Since $\operatorname{rank}\psi(R) = \operatorname{rank} R = \operatorname{rank} UR^*U^*$, we obtain $\psi(R) = UR^*U^*$. Examining the remaining cases concerning U and f, we find that ψ is of one of the following forms:

(i1) there exists a unitary U on H_n such that
$$\psi(R) = URU^* \qquad (R \in PI(H_n));$$

(ii1) there exists a unitary U on H_n such that
$$\psi(R) = UR^*U^* \qquad (R \in PI(H_n));$$

(iii1) there exists an antiunitary U on H_n such that
$$\psi(R) = URU^* \qquad (R \in PI(H_n));$$

(iv1) there exists an antiunitary U on H_n such that
$$\psi(R) = UR^*U^* \qquad (R \in PI(H_n)).$$

Going back to our original map ϕ, we see that on $PI(M,N)$ the function ϕ is of one of the following forms:

(i2) there exist unitaries U, V on H such that
$$\phi(R) = URV \qquad (R \in PI(M,N));$$

(ii2) there exist unitaries U, V on H such that
$$\phi(R) = UR^*V \qquad (R \in PI(M,N));$$

(iii2) there exist antiunitaries U, V on H such that

$$\phi(R) = URV \qquad (R \in PI(M, N));$$

(iv2) there exist antiunitaries U, V on H such that

$$\phi(R) = UR^*V \qquad (R \in PI(M, N)).$$

Pick two mutually orthogonal unit vectors $x, y \in H$ and consider the partial isometries $R_1 = x \otimes x$, $R_2 = ix \otimes x$, $R_3 = x \otimes y$. Denote by $[x, y]$ the subspace generated by x, y. On $PI([x, y], [x, y])$ ϕ is of one of the above forms. Suppose that we have (ii2). It follows that $\phi(R_2) = -i\phi(R_1)$, $\phi(R_3)^*\phi(R_1) = 0$ and $\phi(R_3)\phi(R_1)^* \neq 0$. Consider a finite dimensional subspace H_0 of H which contains all the initial and final spaces of $R_{\alpha_0}, R_1, R_2, R_3$. It is easy to see that on $PI(H_0, H_0)$ the transformation ϕ must be of the form (ii2). In fact, the other possibilities can be easily excluded considering the relations among $\phi(R_1), \phi(R_2)$ and $\phi(R_3)$.

Let us take any finite rank operator $A \in F(H)$. By spectral theorem and polar decomposition, A can be written as a finite sum

$$A = \sum_k \lambda_k P_k, \tag{3}$$

where P_k's are finite rank partial isometries and λ_k's are scalar. We define

$$\Phi(A) = \sum_k \overline{\lambda_k}\phi(P_k).$$

We claim that Φ is a conjugate-linear transformation on $F(H)$ extending the restriction of ϕ onto the set of all finite rank partial isometries on H. First, we have to show that Φ is well-defined. Let

$$A = \sum_l \mu_l Q_l$$

be another resolution of A similar to what was given in (3). Let $H_0 \subset H_1$ be a finite dimensional subspace of H which contains the initial and finial spaces of all P_k, Q_l appearing above. It follows that on $PI(H_1, H_1)$ the transformation ϕ is of the form (ii2). Therefore, we can write

$$\sum_k \overline{\lambda_k}\phi(P_k) = UA^*V = \sum_l \overline{\mu_l}\phi(Q_l)$$

which means that Φ is well-defined. Now, the additivity and conjugate-linearity of Φ is trivial to verify. So, we have a conjugate-linear map $\Phi : F(H) \to F(H)$ which extends the restriction of ϕ onto the set of all finite rank partial isometries. Using the 'local' form (ii2) of ϕ, it is obvious that Φ satisfies $\Phi(AB^*A) = \Phi(A)\Phi(B)^*\Phi(A)$ $(A, B \in F(H))$, that is Φ is a conjugate-linear Jordan-triple homomorphism. Since every finite rank partial isometry is in the range of Φ, it follows that Φ is surjective. If $A \neq 0$, then we can write A in the form (3) where P_k's are pairwise orthogonal finite rank partial isometries. By the orthogonality preserving property of ϕ we readily have $\Phi(A) \neq 0$. Therefore, the transformation $A \mapsto \Phi(A^*)$ is a

linear Jordan-triple automorphism of $F(H)$. Now, we can apply the result [5, Theorem 3] on the form of linear Jordan-triple isomorphisms between standard operator algebras (in that paper we used the term 'triple isomorphism'). This gives us that, on the set of all finite rank partial isometries, ϕ is one of the forms (ii2), (iii2). Using the complete orthoadditivity of ϕ we find that the same holds true on the whole set $PI(H)$.

The last part of the proof has begun with supposing that ϕ is of the form (ii2) on a certain subset $PI(H_0, H_0)$ of the set of all partial isometries. In any of the other cases one can apply very similar arguments. □

Remark. Observe that 0 is an isolated point of $PI(H)$, so the continuity of ϕ at 0 would mean nothing. We remark that it would be a nice improvement to show (if it is true at all) that the result above holds without our assumption on continuity.

Finally, we recall that it is quite common to refer to the analogue result concerning the lattice of projections as the fundamental theorem of projective geometry.

Wigner's celebrated unitary-antiunitary theorem can be formulated in several ways. One possibility is the following (see, for example, [1]). If $\phi : P_1(H) \to P_1(H)$ is a bijective function which preserves the transition probabilities, that is,

$$\operatorname{tr} \phi(P)\phi(Q) = \operatorname{tr} PQ \qquad (P, Q \in P_1(H)),$$

then there is an either unitary or antiunitary operator U on H such that ϕ is of the form

$$\phi(P) = UPU^* \qquad (P, Q \in P_1(H)).$$

Our next result gives an analogue assertion concerning the set of all rank-1 partial isometries. For other Wigner-type results on other structures we refer to our recent papers [3], [4].

Theorem 2. *Let $\phi : PI_1(H) \to PI_1(H)$ be a bijective function with the property that*

$$\operatorname{tr} \phi(P)^*\phi(Q) = \operatorname{tr} P^*Q \qquad (P, Q \in PI_1(H)). \tag{4}$$

Then ϕ is of one of the following forms:

(a) *there exist unitaries U, V on H such that*

$$\phi(R) = URV \qquad (R \in PI_1(H));$$

(b) *there exist antiunitaries U, V on H such that*

$$\phi(R) = UR^*V \qquad (R \in PI_1(H));$$

Proof. As we have noted in the proof of our first theorem, if $A \in F(H)$, then A can be written as a finite sum $A = \sum_k \lambda_k P_k$ where P_k's are rank-one partial isometries and λ_k's are scalars. We define

$$\Phi(A) = \sum_k \lambda_k \phi(P_k).$$

We claim that $\Phi(A)$ is well-defined. Indeed, if $A=\sum_l \mu_l Q_k$ is another resolution of A of the above kind, then we compute

$$\begin{gathered}\operatorname{tr}\phi(R)^*(\sum_k \lambda_k \phi(P_k))=\sum_k \lambda_k \operatorname{tr}\phi(R)^*\phi(P_k)= \\ \sum_k \lambda_k \operatorname{tr} R^* P_k=\operatorname{tr} R^*(\sum_k \lambda_k P_k)=\operatorname{tr} R^* A\end{gathered} \tag{5}$$

and, similarly,

$$\operatorname{tr}\phi(R)^*(\sum_l \mu_l \phi(Q_l))=\operatorname{tr} R^* A.$$

Therefore, it follows that

$$\operatorname{tr}\phi(R)^*(\sum_k \lambda_k \phi(P_k))=\operatorname{tr}\phi(R)^*(\sum_l \mu_l \phi(Q_l))$$

for every $R\in PI_1(H)$. Since ϕ is surjective, we obtain that $\sum_k \lambda_k\phi(P_k)=\sum_l \mu_l\phi(Q_l)$. So, Φ is well-defined. It is now obvious that Φ is a linear transformation on $F(H)$. Since the range of Φ contains every rank-1 partial isometry, it follows that Φ is surjective. An operator $A\in F(H)$ has rank 1 if and only if it is a nonzero scalar multiple of a partial isometry. It follows that Φ preserves the rank-1 operators in both directions. The form of all surjective linear (even additive) transformations on $F(H)$ which preserve the rank-1 operators is known. Namely, it follows from [6, Theorem 3.3] that either there are bijective linear operators U,V on H such that

$$\Phi(x\otimes y)=Ux\otimes Vy \qquad (x,y\in H)$$

or there are bijective conjugate-linear operators U,V on H such that

$$\Phi(x\otimes y)=Uy\otimes Vx \qquad (x,y\in H).$$

Depending on the actual case, we easily obtain from (4) that U,V are scalar multiples of unitaries or antiunitaries. It is now trivial to complete the proof. □

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